Developing a Rubric for Assessing Mathematical Reasoning: A Design-Based Research Study in Primary Classrooms

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Despite mathematical reasoning being a proficiency included in mathematics curricula around the world, research has found that primary teachers struggle to understand, teach, and assess mathematical reasoning. A detailed rubric involving the three reasoning actions of analysing, generalising and justifying at five proficiency levels was refined according to feedback from teachers. At different stages of the study, teachers used the rubric to assess their students' reasoning and provided feedback about its usefulness.

In the Australian Curriculum: Mathematics (Australian Curriculum and Assessment Authority [ACARA], 2017), reasoning is explicitly stated as a proficiency to be developed in students and is defined as being the '… capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising'. Despite its emphasis in many curricula around the world, research on teachers' knowledge and understanding of reasoning indicates that many teachers need support in enacting and assessing many aspects of this proficiency (Blanton and Kaput, 2005; Clarke, Clarke & Sullivan, 2012). This need has resulted in calls for more opportunities for teachers to learn about students' mathematical reasoning and its development (Francisco & Maher, 2011). Our previous research found that teachers developed their knowledge and understanding of reasoning through demonstration lessons and teaching it themselves (Loong, Vale, Herbert, Bragg & Widjaja, 2017). In addition, teachers need to know how students' reasoning can be assessed formatively. This assessment allows teachers to monitor students' reasoning proficiency and further develop it through regular planning of tasks that elicit a variety of reasoning actions, other than a commonly used action like explaining (Clarke, et al., 2012). A rubric for assessing mathematical reasoning will help teachers be aware of the reasoning actions and formatively assess the reasoning articulated and displayed by students. Pegg, Gutiérrez and Huerta (1998) noted that a method of assessment may not fit the specific requirements of teachers for various reasons, for example, it may be too time consuming, require an understanding of the topic or nature of learners' responses not accessible to the teacher, or may not be appropriate for a school context. For this reason, we chose to use a design-based research methodology (Wang & Hannafin, 2005) where teacher participants helped refined the rubric we designed to a level that teachers find useful. This paper reports on our experience in developing a formative assessment rubric for reasoning for reSolve, a national project funded by the Australian Federal Department of Education and Training

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Background

Reasoning Frameworks

Jeannotte and Kieran (2017) identified two categories of reasoning definitions in curriculum statements: the "structural aspect of mathematical reasoning (p. 7)" and the "process aspect of mathematical reasoning" (p.9). It is the "process aspect of reasoning" that is included in the Australian Curriculum. The search for similarities and differences, and processes related to validating are the two main categories of reasoning, where comparing and contrasting objects leads to forming conjectures and generalising. Ellis (2007) identified three levels of comparing and contrasting when analysing Year 7 student responses to growing patterns tasks: 1) relating, 2) searching and 3) extending. Lannin, Ellis and Elliot (2011) reorganised these categories and combined conjecturing and generalising to nominate four essential understandings of generalising: 1) developing statements; 2) identifying commonality and extending beyond original cases; 3) recognising a domain for which the generalisation holds; and 4) "clarifying the meaning of terms, symbols and representations" (p. 12). Validating enables students to convince others that a conjecture or generalisation is justified (Carpenter, Franke & Levi, 2003). Explanations are not sufficient to be convincing. Carpenter et al. (2003) identified three classes of justification to describe the ways in which primary students justify and argue: "appeal to authority; justification by example; and generalisable arguments" (p. 87). The authors tested these categories and levels of reasoning when investigating Year 3-4 students' reasoning when working on a commonality problem. Table 1 displays the levels of reasoning for the three reasoning actions identified in the students' reasoning.

Table 1
Reasoning Actions and Levels: 'What else belongs?' (Vale et al., 2017)

Reasoning Actions	Reasoning Levels
Comparing and contrasting	Noticing (seeing) similarities or relations
	Noticing commonalities and differences
	Searching for commonalities
Generalising	Forming conjectures about common properties
	Extending a common property through further examples
	Generalising properties
Justifying	No justification
	Appealing to authority or others
	Explaining a common property using an example or counter property
	Verifying that the common property holds for each member of the group
	Extending generalisation using logical argument

Assessing Reasoning

The literature reports three types of assessment, namely, assessment for learning (AfL) where information collected from assessments are used to modify teaching and learning (ARG, 2002; William, 2011); assessment of learning (AoL) where achievements are summarised for the purpose of recording and reporting to relevant parties (Harlen, 2007);

assessment as learning (AaL) where the student monitors what they are learning and uses that feedback to make adjustments, adaptations and major changes in what they understand (Earl, 2003). AfL and AaL are formative types of assessment whereas AoL is summative in nature. Formative assessment enables the teacher to systematically gather evidence and provide feedback about learning while instruction is underway. We concur with Pegg and colleagues (1998), on the benefits of "forms of assessment which allow for the interpretation of learners' responses within a framework of cognitive growth… allows teachers to see where their learners are on some developmental ladder and, at the same time, provide advice on possible pathways for future teaching endeavours." (p.4).

The Use of Rubrics for Assessment

Rubrics have been designed and researched for their efficacy in promoting thinking and learning as well as making the assessment criteria required transparent to students (Panadero & Jonsson, 2013). Instructional rubrics have also been found to be useful for teachers. Andrade (2000) for example advocates their use as they help teachers teach, make assessing student work quick and efficient, and help teachers justify to parents and others the grades that they assign to students. Using a rubric allows the teacher to infer the gap between the students' current learning and desired instructional goals, identifying students' emerging understanding or skills so that they can build on these by modifying instruction to facilitate growth.

Rubrics and models that are available for assessing reasoning include those for geometry such as the van Hiele's levels of geometric reasoning (van Hiele, 1986) and the SOLO taxonomy (Pegg et al., 1998). However, these are mostly frameworks that detail the progression of reasoning for a particular content domain rather than for mathematical reasoning in any content domain. Given teachers' need for assessment tools that are pragmatic and usable in schools that tracks cognitive growth in students, our research adopted a design-based methodology to work with teachers to create this rubric.

Methodology

Design-based research is a systematic but flexible methodology that aims to improve educational practices through iterative cycles of design development, implementation, and analysis in collaboration with practitioners (Wang & Hannafin, 2005). Wang and Hannafin identify five basic characteristics of design-based research: pragmatic, grounded, interactive, iterative and flexible, and integrative and contextual. These characteristics are evident in our design process as it seeks to solve a practical on-going issue of how to assess the reasoning proficiency. It is grounded in the theoretical frameworks of teaching and learning of mathematical reasoning and real-world implementations of it. Our design is one where teachers interact with the research team to iteratively refine the rubric. Our research focused on creating a rubric that enables teachers to use it in the everyday context of their teaching. It also aimed to provide teachers with sample lessons that elicit reasoning and examples of teachers' use of the rubric.

Participants

The participants included the six members of the research team, 32 teachers from four primary schools in Victoria, Australia, and a critical friend expert from AAMT/AAS who provided constructive feedback on the rubric in the penultimate stage of project. Two teachers from each year level across Years 3-6 from each of the schools trialled the rubric twice in their grade, each time with the most recent version of the rubric.

Methods

There were four stages in our design-based research project. In Stage 1, the Mathematical Reasoning Research Group (MaRRG) developed an initial reasoning assessment rubric for trial in schools. In Stage 2, the research team provided professional development (PD) on mathematical reasoning to each of the four participating schools to develop teachers' awareness of mathematical reasoning is, and ways to elicit it in their classroom. A one-hour whole school PD workshop on assessing mathematical reasoning was conducted by two researchers at each school prior to teachers trialling the assessing mathematical reasoning rubric. We provided PD because it is crucial in assisting teachers to come to an understanding of the nature of mathematical reasoning (Loong et al., 2017) before attempting to assess it. Researchers met with participating teachers to discuss trialling the reasoning tasks and rubric, and how to use the teacher observation schedule. Pairs of teachers in the same year level selected one of the tasks provided by the research team to teach and observed each other teaching the same lesson. Two researchers observed each lesson. The observing teacher and researchers used the observation sheet to record evidence of student mathematical reasoning. The teaching pair together with the two researchers then engaged in a post-lesson discussion lasting between 30 minutes to one hour. The focus of the post lesson discussion was on teachers assessing student reasoning that they observed, students' work samples and the teacher observation schedule using the rubric. Feedback from the post lesson discussion at the first school led to a modification of the rubric.

In Stage 3, the other three schools trialled Version 2 of the Assessing Reasoning Rubric using one of the tasks provided by the research team. Feedback for further modification occurred successively as each school attempted to use the rubric to assess their children's reasoning. For a second round of trials at each school the researchers then provided links to useful resources for locating tasks with a reasoning focus to support teachers in their development of a follow up lesson intended to include opportunities for students to reason and for teachers to assess. The intention was for them to trial the revised rubric to assess children's reasoning in these tasks. In the second round of Stage 3, classroom teaching and learning of the reasoning task and post lesson discussions with teachers were video-taped to provide data for exemplar materials for the assessment of mathematical reasoning. In Stage 4, the rubric was further revised using feedback from teachers in Stage 3 and the *reSolve* critical friend. The final simplified version was presented to teachers at a mathematics education conference and feedback was gathered using field notes.

Results and Discussion

Stage 1

Our previous research (Vale et al. 2017) provided an initial framework (Table 1) for designing an assessment rubric for mathematical reasoning. However, this framework was based on a particular type of generalisation problem, involving forming conjectures about a common property and was therefore not necessarily appropriate for other generalisation and justification tasks. We investigated other frameworks for other types of reasoning tasks: Lannin et al. (2011) for early algebra problems and Carpenter et al. (2003) for justification and proof tasks. We also consulted NRICH (1997-2018) where five steps were identified in the progression of reasoning proficiency for tasks involving the testing of conjectures. These were:

> *Step one*: Describing: simply tells what they did.
>
> *Step two*: Explaining: offers some reasons for what they did…
>
> *Step three*: Convincing: confident that their chain of reasoning is right…
>
> *Step four*: Justifying: a correct logical argument… uses words such as 'because', 'therefore', 'and so', 'that leads to' ...
>
> *Step five*: Proving: a watertight argument… (https://nrich.maths.org/11336)

Synthesising our findings, we decided upon three key reasoning actions to be included in the initial rubric: 'analysing', 'forming conjectures and generalising' and 'justifying and logical argument.' We found that each of the reasoning verbs included in the definition of reasoning in the Australian curriculum aligned with one of these key reasoning actions. We thought about attempting to identify reasoning outcomes for each year level but the research does not provide evidence for this. Students at a young age are capable of providing a watertight argument relative to their knowledge of content and use of materials and symbols. Conversely, without the opportunity to develop reasoning proficiency students in later years may not have developed the proficiency to notice, generalise and justify. As well, students may display different developmental levels across the three reasoning actions. Consequently, we decided upon five levels in the reasoning learning trajectory for use in the rubric: 'not evident', 'beginning', 'developing', 'consolidating' and 'extending'. Intentionally, the levels were not aligned to school year levels. Figure 1 provides the descriptors in the rubric for the "developing" level for each reasoning action (space does not allow exhibition of all levels for this version).

Stage 2

As a result of feedback from teachers in the post lesson discussions during the first round of the teaching and learning of reasoning tasks at two schools, the following modifications were made. Changes to the rubric included:

Formatting the rubric to fit on a single A4 page.
Providing a space below to include teacher's comment "Evidence of reasoning"
Highlighting/Bolding keywords in the rubric
Reducing "wordiness" of rubric
Consistency of tense and wording in bullet points

Further issues that arose from Stage 2 included the limitations in assessing student reasoning solely on the use of a work sample. Many teachers commented on the ways students often expressed their reasoning verbally and through gesture. They suggested ways teachers could capture this evidence to complement the work sample to provide a more accurate assessment of students' reasoning actions/capabilities. Modifications were made to the rubric to include space for teachers to include evidence of students' gestures and verbal explanations, as listed above and teachers used a revised version in Round 2 of the trials.

	Analysing	Forming Conjectures and Generalising	Justifying and Logical argument
Developing	• Notices a common numerical or spatial property. • Sorts and classifies cases according to a common property. • Orders cases to show what is the same or stays the same and what is different or changes. • Recalls, repeats and extends patterns using numerical structure or spatial structure. • Describes the case or pattern by labelling the category or sequence.	• Communicates a rule about a ***property*** using words, diagrams or number sentences. • Communicate a rule about a ***pattern*** using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition. • Explain the meaning of the rule using one example.	• Checks the truth of statements using materials and informal methods. • Uses known facts to verify that the statement, common property, or rule for a pattern holds for each case. • Uses a counter example to refute a claim. • Starting statements in a logical argument is correct and accepted by the classroom • Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods.

Figure 1. Excerpt from Assessing Mathematical Reasoning Rubric - Version 1.

Stage 3

The first two schools and teachers were observed again using tasks that they had found themselves and schools 3 and 4 were observed twice using tasks provided by the team and those selected by the teachers. All teachers in this stage provided feedback on Version 2 of the rubric. They made positive comments in relation to the modifications made and the overall “user-friendliness” of the rubric, for example, bolding of words, one-page format, reduced wordiness.

> Well the first improvement is it’s all on one page… it’s much more user friendly when it’s all on one page. (School A, Year 5/6 teacher, Round 2)

> I like the use of bold. (School A, Year 5/6 teacher, Round 2)

Overall it is a simpler and less wordy rubric yet still provides support to teachers with less well-developed understanding of reasoning who may require more complete explanations and guidance how to move forward in their planning for reasoning.

> … I quite like it because it’s making me learn what they should be doing, I'm thinking maybe I should be encouraging them to verify the truth of what they’re saying more … (School B, Year 5 teacher, Round 2,)

> … if we can use it as not only to inform us about our students but for where to next. (School A, Year 5/6 teacher, Round 2)

However, feedback from the teachers at these schools also indicated that for teachers with a good understanding of mathematical reasoning, the rubric was still too wordy. Although they have not come across a reasoning rubric, they preferred a simple version. For example:

> It has to be simplified there are too many aspects … it is too time consuming … (School B, Year 4 teacher, Round 2)

However, for teachers still coming to terms with the nature of mathematical reasoning a detailed rubric was useful and provided much guidance and language to use in reports.

> But the statements would be good for us as well to put in reports and stuff, I haven’t done as much for maths reasoning because I’ve always thought it’s more of a high school thing and I haven’t

really thought about it being in primary, but now I'm realising it can…I could use that language to help write reports and stuff of what they need to do next (School B Year 5 Round 2)

There did appear to be some ongoing confusion between formative and summative assessment. Some wanted year level reasoning outcome statements. Teachers tended to view positively assessment that has a summative slant. The following teacher put it this way:

> You could almost have it as a summative assessment cos rather than having not evident, beginning, developing change it to language of F, 1, 1.5, 2, 2.5. We have it in our brains, makes it more practical like the Vic Curric. … (School B, Year 4 teacher, Round 1)

Stage 4

Our critical friend provided pertinent insights into the clarity of the rubric and possible improvements to it. This feedback together with the feedback from teachers in Stage 3 enabled us to make further changes to the rubric condensing it to a page with simple dot points for teachers to use as well as record observations and feedback. Figure 2 is the simplified version. We simplified the headings for the three key reasoning actions: analysing, generalising and justifying. This final version was presented to 35 teachers at a mathematics education conference and feedback from the teachers was positive.

Assessing Mathematical Reasoning Rubric

Student Name: **Reasoning Task:** **Date:**

Observation of student's reasoning:

	Analysing	Generalising	Justifying
Not Evident	• **Does not notice** common property or pattern.	• **Does not communicate** a common property or rule (conjecture).	• **Does not justify**.
Beginning	• **Recalls** random known facts or **attempts to sort** examples or **repeats** patterns.	• **Attempts to communicate** a common property or rule (conjecture) for the pattern.	• **Describes** what they did and **recognises** what is correct or incorrect. • **Argument** is not coherent or does not include all steps.
Developing	• **Notices** a common property, or **sorts and orders** cases, or **repeats and extends** patterns. • **Describes** the property or pattern.	• **Generalises: communicates** a rule (conjecture) using mathematical terms and records other cases or examples.	• **Attempts to verify** by testing cases and **detects and corrects** errors or inconsistencies. • Starting statements in a **logical argument** are correct.
Consolidating	• **Systematically searches** for examples, extends pattern or analyses structure to form a conjecture. • Makes **predictions** about other cases.	• **Generalises: communicates** a rule (conjecture) using mathematical symbols and **explains what** the rule means or **explains how** the rule works using examples.	• **Verifies** truth of statements by confirming all cases or **refutes** a claim by using a counter example. • Uses a correct **logical argument.**
Extending	• **Notices and explores** relationships between properties.	• **Generalises** cases, patterns or properties using mathematical symbols and **applies** the rule. • **Compares** different expressions for the same pattern or property to show equivalence.	• Uses a **watertight logical argument**. • **Verifies** that the generalisation holds for *all* cases using logical argument.

Comments (feedback, reasoning prompts for further development):

Figure 2. A simplified Assessing Mathematical Reasoning Rubric.

Conclusion

This paper described a design-based research study that drew upon the expertise and experience of teachers to refine the assessment of reasoning rubric. Teachers' feedback was valuable in refining the rubric to provide sufficient detail for teachers to understand what mathematical reasoning is and what to look for when assessing children's reasoning. Whilst teachers who are confident in their knowledge and understanding of mathematical reasoning felt that a more useful rubric would be a summative rubric, the detail in the rubric was helpful for teachers who needed further development in the area of developing reasoning in students and in reporting student progress. Setting up the rubric with developmental stages and descriptions for each reasoning action provided insights into each of the three reasoning actions as individual learning trajectories. Teachers will be provided with both the detailed and simplified versions of the rubric. How pragmatic and successful the rubric is in meeting the needs of primary teachers remains to be seen. Follow up research might reveal if this is a useful tool for teachers.

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